Mock Exam Analysis on Manifolds

March, 2015

Assignment 1. (20 pt.)
We consider smooth differential forms on $\mathbb{R}^3$.

1. Prove that the one-form $\sigma = yz\,dx + xz\,dy + xy\,dz$ is exact, and determine a function f such that $\sigma = df$.

2. Prove that the 3-form
$$\omega = xyz\,dx \wedge dy \wedge dz \tag{1}$$
on $\mathbb{R}^3$ is exact, and determine a two-form η on $\mathbb{R}^3$ such that $\omega = d\eta$.

3. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation to cylindrical coordinates given by
$$\phi(r, \theta, z) = (r\cos\theta, r\sin\theta, z).$$
Determine $\phi^*\omega$, with ω given by (1).

Assignment 2. (20 pt.)
Let M be the set of lines in $\mathbb{R}^2$.

1. Prove that M is a two-dimensional C^∞–manifold.
(Hint: construct parametrizations of the set L_1 of non-vertical lines and the set L_2 of non-horizontal lines.)

2. Prove that every translation on $\mathbb{R}^2$ induces a C^∞–diffeomorphism on M.

Assignment 3. (25 pt.)
The 2-vorm ω on $\mathbb{R}^3 \setminus \{(0,0,0)\}$ is given by
$$\omega = \frac{1}{(x^2+y^2+z^2)^{3/2}}\,(x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy).$$

1. Prove that ω is closed.

2. Prove that $\int_{\mathbb{S}^2} \omega \neq 0$.

   Hint: note that, on $\mathbb{S}^2$, the 2-form ω is equal to the 2-form η given by
$$\eta = x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy.$$

3. Prove that ω is not exact.

Z.O.Z.

Assignment 4. (25 pt.)
Let M be a two-dimensional Riemannian manifold, and let $\{F_1, F_2\}$ be a moving frame on a subset V of M, with connection form ω_{12}.

1. At points of V the Gaussian curvature is given by
$$K = F_2(\omega_{12}(F_1)) - F_1(\omega_{12}(F_2)) - \omega_{12}(F_1)^2 - \omega_{12}(F_2)^2.$$
(Hint: express ω_{12} in terms of the coframe $\{\vartheta_1, \vartheta_2\}$. See also the hint at the end of this exercise.)

2. Prove that the Lie-bracket of F_1 and F_2 is given by
$$[F_1, F_2] = -\omega_{12}(F_1)\, F_1 - \omega_{12}(F_2)\, F_2.$$
(Hint: you may want to determine $\vartheta_1([F_1, F_2])$ and $\vartheta_2([F_1, F_2])$ using the identities given at the end of this exercise.)

3. Let M be the upper half plane $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$ endowed with the Riemannian metric $\langle \cdot, \cdot \rangle$, defined by
$$\langle v, w \rangle = \frac{v \cdot w}{x_2^2},$$
for $v, w \in T_{(x_1,x_2)}M$, where $v \cdot w$ is the standard inner product on $\mathbb{R}^2$. Let
$$F_1 = x_2 \frac{\partial}{\partial x_1}, \qquad F_2 = x_2 \frac{\partial}{\partial x_2}.$$
Use part 2 to determine the connection form of the moving frame $\{F_1, F_2\}$.

4. Prove that M has constant Gaussian curvature $K = -1$.

Hint: Recall the Levi-Civita equations
$$\begin{aligned} d\vartheta_1 &= \omega_{12} \wedge \vartheta_2, \\ d\vartheta_2 &= -\omega_{12} \wedge \vartheta_1. \end{aligned}$$
You may also want to use the identity
$$\sigma([X, Y]) = X(\sigma(Y)) - Y(\sigma(X)) - d\sigma(X, Y),$$
for a one-form σ and vector fields X and Y.

Solutions

Assignment 1 (7+6+7 pt.)

1. Use Poincaré's Lemma to show that ω is exact by proving that $d\omega = 0$. This follows from a straightforward computation. The function f for which $\sigma = df$ has to satisfy the equations

$$\frac{\partial f}{\partial x} = yz, \qquad \frac{\partial f}{\partial y} = xz, \qquad \frac{\partial f}{\partial z} = xy.$$

Take, e.g., $f(x, y, z) = xyz$.

2. Every 3-form on $\mathbb{R}^3$ is closed. By Poincaré's Lemma ω is exact. Note that $\omega = d(\frac{1}{2}x^2yz) \wedge dy \wedge dz$, so $\omega = d\eta$ for $\eta = \frac{1}{2}x^2yz \wedge dy \wedge dz$.

3.

$$\begin{aligned} \varphi^*(\omega) &= (x \circ \varphi)(y \circ \varphi)(z \circ \varphi)d(x \circ \varphi) \wedge d(y \circ \varphi) \wedge d(z \circ \varphi) \\ &= r^3 z \cos\vartheta \sin\vartheta \, dr \wedge d\vartheta \wedge dz. \end{aligned}$$

Assignment 2 (10+10 pt.)

1. Let $U_i = \mathbb{R}^2$, let $L_1 \subset M$ be the set of non-vertical lines, and let $L_2 \subset M$ be the set of non-horizontal lines in $\mathbb{R}^2$. Let $f_1 : U_1 \to M$ map the point (u_1, u_2) to the line with equation $y = u_1x + u_2$, and let $f_2 : U_2 \to M$ map the point (u_1, u_2) to the line with equation $x = u_1y + u_2$. Then f_i is a homeomorphism $U_i \to L_i$. Furthermore, $U_{12} := f_i^{-1}(L_1 \cap L_2) = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_1 \neq 0\}$. The map $f_2^{-1} \circ f_1 : U_{12} \to U_{12}$ is given by

$$f_2^{-1} \circ f_1(u_1, u_2) = (\frac{1}{u_1}, -\frac{u_2}{u_1}).$$

Since this map is smooth (C^∞), the charts (U_1, f_1) and (U_2, f_2) define a C^∞-structure on M.

2. Let $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ be the translation given by$\psi(x, y) = (x + a, y + b)$. Since ψ maps lines to lines in a 1-1 way, it induces a bijection on M. We prove that this bijection is differentiable at a point $l \in L_1$. Let $l = f_1(u_1, u_2)$, then $(x, y) \in \psi(l)$ iff $\psi^{-1}(x, y) \in l$, i.e., iff $y - b = u_1(x - a) + u_2$. Therefore, $\psi(l)$ is the line with equation $y = u_1x + u_2 - u_1a + b$, i.e.,

$$f_1^{-1} \circ \psi \circ f_1(u_1, u_2) = (u_1, u_2 - u_1b + a).$$

Since this map is differentiable, ψ is differentiable at all points of L_1. A similar argument shows that ψ is differentiable at all points of L_2, so ψ is differentiable on M. Sine ψ^{-1} is also a translation, we conclude that ψ is a diffeomorphism on M.

Assignment 3 (8+9+8 pt).

1. A straightforward calculation shows that $d\omega = 0$.

2. Using the hint we derive

$$\begin{aligned}\int_{\mathbb{S}^2} \omega &= \int_{\partial\mathbb{B}^3} \eta \\ &= \int_{\mathbb{B}^3} d\eta \\ &= 3 \int_{\mathbb{B}^3} dx \wedge dy \wedge dz \\ &= 3 \text{ Volume}(\mathbb{B}^3) \\ &= 4\pi \\ &\neq 0.\end{aligned}$$

3. Assume ω is exact, say $\omega = d\sigma$ for a 1-form σ on $\mathbb{R}^3 \setminus \{(0,0,0)\}$. Then $\int_{\mathbb{S}^2} \omega = \int_{\mathbb{S}^2} d\sigma = \int_{\partial\mathbb{S}^2} \sigma = 0$, because $\partial\mathbb{S}^2 = \emptyset$.

Assignment 4 (8+8+5+4 pt.)

1. Let $\omega_{12} = f_1\vartheta_1 + f_2\vartheta_2$, i.e., let $f_i = \omega_{12}(F_i)$. Then

$$d\omega_{12} = df_1 \wedge \vartheta_1 + f_1\, d\vartheta_1 + df_2 \wedge \vartheta_2 + f_2\, d\vartheta_2. \tag{2}$$

Using the Levi-Civita equations we get

$$\begin{aligned}d\vartheta_1 &= f_1\vartheta_1 \wedge \vartheta_2, \\ d\vartheta_2 &= f_2\vartheta_1 \wedge \vartheta_2.\end{aligned}$$

Furthermore,

$$df_i = df_i(F_1)\,\vartheta_1 + df_i(F_2)\,\vartheta_2 = F_1(f_i)\,\vartheta_i + F_2(f_i)\,\vartheta_2.$$

Substituting the expressions for $d\vartheta_1$, $d\vartheta_2$ and df_i into (2), we get

$$d\omega_{12} = (-F_2(f_1) + f_1^2 + F_1(f_2) + f_2^2)\,\vartheta_1 \wedge \vartheta_2.$$

Since K is uniquely determined by $d\omega_{12} = -K\vartheta_1 \wedge \vartheta_2$, we obtain the requested expression for K.

2. We have to prove that $\vartheta_i([F_1, F_2]) = -\omega_{12}(F_i)$. Using the identity at the end of the assignment with $\sigma = \vartheta_i$, $X = F_1$ and $Y = F_2$, we get

$$\begin{aligned}\vartheta_i([F_1,F_2]) &= -d\vartheta_i(F_1,F_2) + F_1(\vartheta_i(F_2)) - F_2(\vartheta_i(F_1)) \\ &= -d\vartheta_i(F_1,F_2) + F_1(\delta_{i2}) - F_2(\delta_{i1}) \\ &= -d\vartheta_i(F_1,F_2).\end{aligned}$$

Since $d\vartheta_1(F_1,F_2) = (\omega_{12} \wedge \vartheta_i)(F_1,F_2) = \omega_{12}(F_1)$, we see that $\vartheta_1([F_1,F_2]) = -\omega_{12}(F_1)$. In a similar way we show that $\vartheta_2([F_1,F_2]) = -\omega_{12}(F_2)$.

3. A straightforward computation shows that $[F_1, F_2] = -F_1$:

$$[F_1, F_2](f) = x_2 \frac{\partial}{\partial x_1}(x_2 \frac{\partial f}{\partial x_2}) - x_2 \frac{\partial}{\partial x_2}(x_2 \frac{\partial f}{\partial x_1}) = -x_2 \frac{\partial f}{\partial x_1} = -F_1(f).$$

Therefore, $\omega_{12}(F_1) = 1$ and $\omega_{12}(F_2) = 0$, so $\omega_{12} = \vartheta_1$.

4. Using the result of part 3 we get $K = F_2(1) - F_1(0) - 1^2 - 0^2 = -1$.